Problem Set - 3 (Random Variables and Distributions)

ECO 104 (Section 8)
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Due Date : 17th Dec, 2023, Sunday 10:00 AM (submit in Google Class), individual assignment

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You should write the solutions neatly in paper and submit in class. **This is NOT a group assignment, this is an individual assignment**, so please do the problems and submit individually! Please ask me in Ed if you have any question.

§. Problems from Discrete Random Variables

1. From Anderson et al. (2020) Chapter 5, #1

   Consider the experiment of tossing a coin twice.

   (a) List the experimental outcomes.

   (b) Define a random variable that represents the number of heads occurring on the two tosses.

   (c) Show what value the random variable would assume for each of the experimental outcomes.

   (d) Is this random variable discrete or continuous?

2. From Anderson et al. (2020) Chapter 5, #2

   Consider the experiment of a worker assembling a product.

   (a) Define a random variable X that represents the time in minutes required to assemble the product.

   (b) What values may X take?

   (c) Is X a discrete or continuous random variable?

3. From Anderson et al. (2020) Chapter 5, #7.

   The probability mass function or PMF for the random variable X follows.

x	$f(x)$
20	0.20
25	0.15
30	0.25
35	0.40

   (a) Is this a valid PMF? Explain.

   (b) What is $\mathbb{P}(X = 30)$?

   (c) What is $\mathbb{P}(X \leq 25)$?

   (d) What is $\mathbb{P}(X \leq 30)$?

4. From Anderson et al. (2020) Chapter 5, #15.

   The following table provides a PMF for the random variable X.

x	$f(x)$
3	0.25
6	0.50
9	0.25

   (a) Compute $\mathbb{E}(X)$ and $\mathbb{V}\text{ar}(X)$.

   (b) Compute the standard deviation of X.

   (c) What will happen to variance if we change the distribution as follows,

x	$f(x)$
3	0.3333
6	0.3333
9	0.3333

5. From Anderson et al. (2020) Chapter 5, #16 (slightly modified)

   The following table provides a probability distribution for the random variable Y,

y	$f(y)$
2	0.20
4	0.30
7	0.40
8	0.10

   (a) Compute $\mathbb{E}(Y)$ and $\mathbb{V}\mathrm{ar}(Y)$.

   (b) Compute $\mathbb{E}(Y^2)$. (To do this you can apply the expectation formula for Y^2. In general for a function $g(Y)$, we can do $\mathbb{E}(g(Y)))$ [This is called Law of Unconscious Statistician (LOTUS)

   (c) Check whether $\mathbb{V}\mathrm{ar}(Y) = \mathbb{E}(Y^2) - [\mathbb{E}(Y)]^2$. Can you prove this relation? (Hint: Use the definition of $\mathbb{V}\mathrm{ar}(Y)$)

6. From Anderson et al. (2020) Chapter 5, #31.

   Consider a Binomial experiment with 2 trials and $p = .4$. This means we have a random variable X such that $X \sim \mathrm{Bin}(2, 0.4)$

   (a) Compute $f(0), f(1), f(2)$ and interpret the value.

   (b) Compute the probability of at least one success, this means $\mathbb{P}(X \geq 1)$

   (c) Compute $\mathbb{E}(X)$ and $\mathbb{V}\mathrm{ar}(X)$.

7. From Anderson et al. (2020) Chapter 5, #32.

   Consider a Binomial experiment with $n = 10$ and $p = .10$. This means we have a random variable $X \sim \mathrm{Bin}(10, 0.10)$

   (a) Compute $f(0)$.

   (b) Compute $f(2)$.

   (c) Compute $\mathbb{P}(X \leq 2)$.

   (d) Compute $\mathbb{P}(X \geq 1)$.

   (e) Compute $\mathbb{E}(X)$ and $\mathbb{V}\mathrm{ar}(X)$.

8. From Anderson et al. (2020) Chapter 5, #33.

   Consider a binomial experiment with $n = 20$ and $p = .70$. This means we have a random variable $X \sim \mathrm{Bin}(20, 0.70)$

   (a) Compute $f(12)$.

   (b) Compute $f(16)$.

   (c) Compute $\mathbb{P}(X \geq 16)$.

   (d) Compute $\mathbb{P}(X \leq 15)$.

   (e) Compute $\mathbb{E}(X)$ and $\mathbb{V}\mathrm{ar}(X)$.

§. **Applied Problems** Solve following applied problems from Anderson et al. (2020)

9. From Anderson et al. (2020) Chapter 5, #36.

   Number of Defective Parts. When a new machine is functioning properly, only 3% of the items produced are defective. Assume that we will randomly select two parts produced on the machine and that we are interested in the number of defective parts found.

   (a) How can you think about a Bernoulli random variable here?

   (b) How can you think about a Binomial random variable here? And what is the key condition under which we can think about a Binomial random variable?

   (c) What are the possible values of the Binomial random variable?

   (d) What is the mean and variance of the Binomial random variable?

10. From Anderson et al. (2020) Chapter 5, #41.

    Introductory Statistics Course Withdrawals. A university found that 20% of its students withdraw without completing the introductory statistics course. Assume that 20 students registered for the course.

(a) Compute the probability that 2 or fewer will withdraw.
(b) Compute the probability that exactly 4 will withdraw.
(c) Compute the probability that more than 3 will withdraw.
(d) Compute the expected number of withdrawals.

11. From Anderson et al. (2020) Chapter 5, #42.

State of the Nation Survey. Suppose a sample of 20 Americans is selected as part of a study of the state of the nation. The Americans in the sample are asked whether or not they are satisfied with the way things are going in the United States.

(a) Compute the probability that exactly 4 of the 20 Americans surveyed are satisfied with the way things are going in the United States.
(b) Compute the probability that at least 2 of the Americans surveyed are satisfied with the way things are going in the United States.
(c) For the sample of 20 Americans, compute the expected number of Americans who are satisfied with the way things are going in the United States.
(d) For the sample of 20 Americans, compute the variance and standard deviation of the number of Americans who are satisfied with the way things are going in the United States.

§. Problems from Continuous Random Variables

12. From Anderson et al. (2020) Chapter 6, #1

The random variable X is known to be uniformly distributed between 1.0 and 1.5 (**in notation we write** $X \sim \mathcal{U}_{[1,1.5]}$**, or we can also write** $X \sim$ **Unif**$(1, 1.5)$, they both mean same thing!)

(a) Show the graph of the probability density function.
(b) Compute $\mathbb{P}(X = 1.25)$.
(c) Compute $\mathbb{P}(1.0 \leq X \leq 1.25)$.
(d) Compute $\mathbb{P}(1.20 < X < 1.5)$.

13. From Anderson et al. (2020) Chapter 6, #2

The random variable X is known to be uniformly distributed between 10 and 20 (in notation we write $X \sim \mathcal{U}_{[10,20]}$)

(a) Show the graph of the probability density function.
(b) Compute $\mathbb{P}(X < 15)$ or Calculate $F(15)$.
(c) Compute $\mathbb{P}(12 \leq X \leq 18)$.
(d) Compute $\mathbb{E}(X)$.
(e) $\mathrm{Var}(X)$.

14. From Anderson et al. (2020) Chapter 6, #9

A random variable X is normally distributed with a mean of $\mu = 50$ and a standard deviation of $\sigma = 5$ (in notation we write $x \sim \mathcal{N}(50, 25)$)

(a) Sketch a normal curve for the probability density function (Using Figure 6.6 in Anderson et al. (2020) as a guide). Label the horizontal axis with values of $35, 40, 45, 50, 55, 60,$ and 65.
(b) What is the probability the random variable will assume a value between 45 and 55? This means $\mathbb{P}(45 < X < 55) =?$
(c) What is the probability the random variable will assume a value between 40 and 60. This means $\mathbb{P}(40 < X < 60) =?$

15. From Anderson et al. (2020) Chapter 6, #10

Suppose we have a random variable which is distributed as a standard normal distribution (this means $Z \sim \mathcal{N}(0, 1)$. Label the horizontal axis at values of $-3, -2, -1, 0, 1, 2,$ and 3. Then use the table in Anderson et al. (2020) to compute the following probabilities.

(a) $\mathbb{P}(Z \leq 1.5)$
(b) $\mathbb{P}(Z \leq 1)$

(c) $\mathbb{P}(1 \leq Z \leq 1.5)$

(d) $\mathbb{P}(0 < Z < 2.5)$

16. From Anderson et al. (2020) Chapter 6, #12. Given that $Z \sim \mathcal{N}(0,1)$, compute the following probabilities.
    (a) $\mathbb{P}(0 \leq Z \leq .83)$
    (b) $\mathbb{P}(-1.57 \leq Z \leq 0)$
    (c) $\mathbb{P}(Z > .44)$
    (d) $\mathbb{P}(Z \geq -.23)$
    (e) $\mathbb{P}(Z < 1.20)$
    (f) $\mathbb{P}(Z \leq -.71)$

17. From Anderson et al. (2020) Chapter 6, #15.Given that $Z \sim \mathcal{N}(0,1)$, find z for each situation.
    (a) The area to the left of z is $.2119$.
    (b) The area between $-z$ and z is $.9030$.
    (c) The area between $-z$ and z is $.2052$.
    (d) The area to the left of z is $.9948$.
    (e) The area to the right of z is $.6915$.

18. (**SKIP THIS FOR EXAM**) From Anderson et al. (2020) Chapter 6, #33.

    Consider a random variable following exponential probability density function.

$$f(x) = \frac{1}{3}e^{-x/3} \quad \text{for } x \geq 0$$

    (a) Write the formula for $P\left(x \leq x_0\right)$.
    (b) Find $\mathbb{P}(X \leq 2)$.
    (c) Find $\mathbb{P}(X \geq 3)$.
    (d) Find $\mathbb{P}(X \leq 5)$.
    (e) Find $\mathbb{P}(2 \leq X \leq 5)$.

§. **Applied Problems** Here are some applied problems from Anderson et al. (2020)

19. From Anderson et al. (2020) Chapter 6, #3.

    Cincinnati to Tampa Flight Time. Delta Airlines quotes a flight time of 2 hours, 5 minutes for its flights from Cincinnati to Tampa. Suppose we believe that actual flight times are uniformly distributed between 2 hours and 2 hours, 20 minutes.
    (a) Show the graph of the probability density function for flight time.
    (b) What is the probability that the flight will be no more than 5 minutes late?
    (c) What is the probability that the flight will be more than 10 minutes late?
    (d) What is the expected flight time?

20. From Anderson et al. (2020) Chapter 6, #7.

    Bidding on Land. Suppose we are interested in bidding on a piece of land and we know one other bidder is interested. The seller announced that the highest bid in excess of $\$10,000$ will be accepted. Assume that the competitor's bid x is a random variable that is uniformly distributed between $\$10,000$ and $\$15,000$.
    (a) Suppose you bid $\$12,000$. What is the probability that your bid will be accepted?
    (b) Suppose you bid $\$14,000$. What is the probability that your bid will be accepted?
    (c) What amount should you bid to maximize the probability that you get the property?

21. From Anderson et al. (2020) Chapter 6, #17.

    Height of Dutch Men. Males in the Netherlands are the tallest, on average, in the world with an average height of 183 centimeters (cm) (BBC News website). Assume that the height of men in the Netherlands is normally distributed with a mean of 183 cm and standard deviation of $10.5\ \mathrm{cm}$.
    (a) What is the probability that a Dutch male is shorter than $175\ \mathrm{cm}$?
    (b) What is the probability that a Dutch male is taller than $195\ \mathrm{cm}$?

(c) What is the probability that a Dutch male is between 173 and 193 cm ?

(d) Out of a random sample of 1000 Dutch men, how many would we expect to be taller than 190 cm ?

22. From Anderson et al. (2020) Chapter 6, #20.

    Gasoline Prices. Suppose that the average price for a gallon of gasoline in the United States is $\$3.73$ and in Russia is $\$3.40$. Assume these averages are the population means in the two countries and that the probability distributions are normally distributed with a standard deviation of $\$.25$ in the United States and a standard deviation of $\$.20$ in Russia.

    (a) What is the probability that a randomly selected gas station in the United States charges less than $\$3.50$ per gallon?

    (b) What percentage of the gas stations in Russia charge less than $\$3.50$ per gallon?

    (c) What is the probability that a randomly selected gas station in Russia charged more than the mean price in the United States?

23. From Anderson et al. (2020) Chapter 6, #19.

    Automobile Repair Costs. Automobile repair costs continue to rise with an average 2015 cost of $367 per repair (U.S. News & World Report website). Assume that the cost for an automobile repair is normally distributed with a standard deviation of $\$88$. Answer the following questions about the cost of automobile repairs.

    (a) What is the probability that the cost will be more than $\$450$?

    (b) What is the probability that the cost will be less than $\$250$?

    (c) What is the probability that the cost will be between $\$250$ and $\$450$?

    (d) If the cost for your car repair is in the lower 5% of automobile repair charges, what is your cost?

24. (**SKIP THIS FOR EXAM**) From Anderson et al. (2020) Chapter 6, #34.

    Phone Battery Life. Battery life between charges for a certain mobile phone is 20 hours when the primary use is talk time, and drops to 7 hours when the phone is primarily used for Internet applications over a cellular network. Assume that the battery life in both cases follows an exponential distribution.

    (a) Show the probability density function for battery life for this phone when its primary use is talk time.

    (b) What is the probability that the battery charge for a randomly selected phone will last no more than 15 hours when its primary use is talk time?

    (c) What is the probability that the battery charge for a randomly selected phone will last more than 20 hours when its primary use is talk time?

    (d) What is the probability that the battery charge for a randomly selected phone will last no more than 5 hours when its primary use is Internet applications?

25. (**SKIP THIS FOR EXAM**) From Anderson et al. (2020) Chapter 6, #38.

    Boston 911 Calls. The Boston Fire Department receives 911 calls at a mean rate of 1.6 calls per hour (Mass.gov website). Suppose the number of calls per hour follows a Poisson probability distribution.

    (a) What is the mean time between 911 calls to the Boston Fire Department in minutes?

    (b) Using the mean in part (a), show the probability density function for the time between 911 calls in minutes.

    (c) What is the probability that there will be less than one hour between 911 calls?

    (d) What is the probability that there will be 30 minutes or more between 911 calls?

    (e) What is the probability that there will be more than 5 minutes, but less than 20 minutes between 911 calls?

Remarks: Many problems are taken from Anderson et al. (2020). If possible you should do more problems from there.

References:

Anderson, D. R., Sweeney, D. J., Williams, T. A., Camm, J. D., Cochran, J. J., Fry, M. J. and Ohlmann, J. W. (2020), *Statistics for Business & Economics*, 14th edn, Cengage, Boston, MA.